Review Lecture Continuum Mechanics

Uwe Mühlich

5. September 2012

Preliminaries

What do we observe?

Objects moving in space, changing their volumes and shapes.

Let's focus on one object. Let's call it "material body"

A body moving in space, changing its volume and shape.

What questions arise?

- What causes a body at rest to move?
- What causes a moving body to stop?
- What causes the change of direction of a moving body?
- *What causes the change of shape / volume of a body?*
- ...

Cause $\Leftrightarrow$ Effect

How to find answers?

- Quantify the effects.
- Come up with a suitable concept for the cause.
- Relate cause(s) and effect(s).

What basic concepts do we need?

Space Time Observer

Questions related to Space, Time, etc.

- most fundamental
- most interesting **and**
- most frightening

questions in physics.

Introductory course in Continuum Mechanics

$\Longrightarrow$ most primitive concepts used here

Space - Time

Definition (Euclidean space $\mathcal{E}$)

- a point set $\mathcal{M}_p$ (e.g. $\mathcal{M}_p = \mathcal{R} \times \mathcal{R}$)
- a vector space $\mathcal{V}$
- a mapping $F : \mathcal{M}_p \times \mathcal{M}_p \to \mathcal{V}$
  assigning a vector to two space points

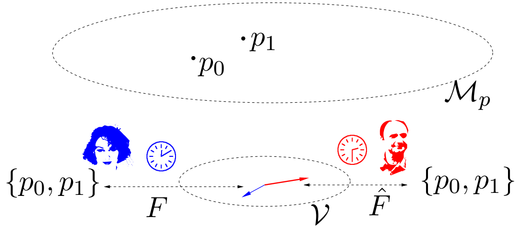
Definition (Time)

Time is used here just as a scalar parameter (some real number) to express the temporal order of events.

Observer

Definition (Observer)

Different observers use different mappings F and clocks.



How can / should  and  compare their measurements?[1]

[1] Revise section "6.1 Change of observer" in your notes!

Body, Configuration and Motion

Material body

Most general definition $\Rightarrow$ body is a differentiable manifold.

Here, a less rigorous definition is used.

Definition (Material body $\mathcal{B}$)

- point set
- points carry physical information (e.g. temperature)
- continuous (smooth) fields can be defined over $\mathcal{B}$

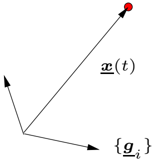
How to quantify the motion of a material body?

Let's start with the motion of "one material point".

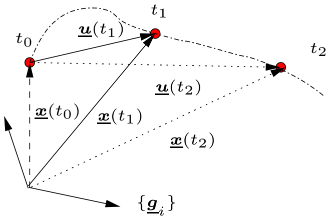
Definition (Motion of a material point)

Change of spatial position in time

Motion of "one material point" (remember basic mechanics)

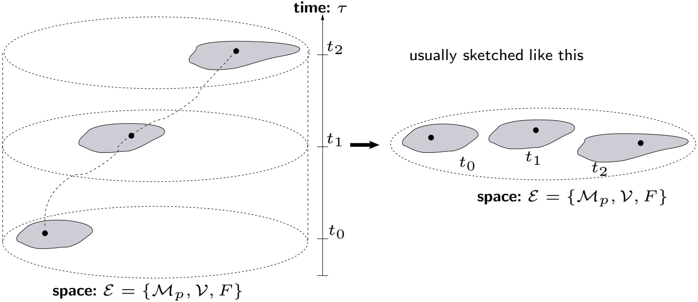


- $\underline{\boldsymbol{x}}(t)$: position of "**the point**" at $\tau = t$
- single measurement not sufficient to decide if there is motion
- reference needed



- one can pick any position as reference
- e.g. initial position ($\tau = t_0$)

Motion of a material body



- There is no longer "**the point**" but there are many points.
- "**the position**" of the body is not precise enough.
- How to distinguish between different points?

Configuration

$\{$Point, Position$\} \Rightarrow \{$Body, Configuration$\}$

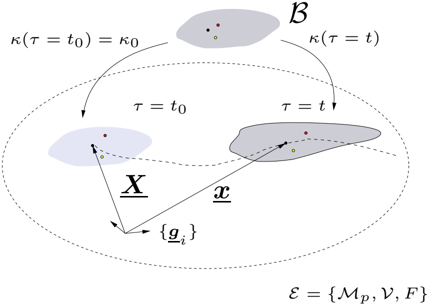
Definition (Configuration)

The configuration of a material body $\mathcal{B}$ at time $\tau = t$ is the region in space occupied by this body at time $\tau = t$.

Material body $\mathcal{B}$ becomes apparent to us only by its configurations.

Labelling of the material points forming $\mathcal{B}$

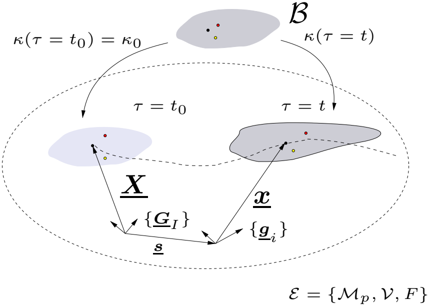
How to distinguish between the different material points?



- pick a reference configuration
- e.g. configuration at $\tau = t_0$
- Every material point is labelled by its position vector in the reference configuration, $\underline{\boldsymbol{X}}$.
- spatial position of a material point with label $\underline{\boldsymbol{X}}$ at time τ $\Rightarrow \underline{\boldsymbol{x}} = \underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, \tau)$

$$\underline{\boldsymbol{X}} = X^1 \underline{\boldsymbol{g}}_1 + X^2 \underline{\boldsymbol{g}}_2 + X^3 \underline{\boldsymbol{g}}_3 = X^i \underline{\boldsymbol{g}}_i \quad | \quad \underline{\boldsymbol{x}} = x^1 \underline{\boldsymbol{g}}_1 + x^2 \underline{\boldsymbol{g}}_2 + x^3 \underline{\boldsymbol{g}}_3 = x^i \underline{\boldsymbol{g}}_i$$

Aim: To distinguish by means of base vectors and indices, if a quantity is defined wrt. the reference configuration or wrt. the actual configuration.



Remarks:

- $\underline{\boldsymbol{X}} = X^I \underline{\boldsymbol{G}}_I$, $\underline{\boldsymbol{x}} = x^i \underline{\boldsymbol{g}}_i$
- usually $\underline{\boldsymbol{s}} = \underline{\boldsymbol{0}}$ and $\underline{\boldsymbol{g}}_i = \delta_i^J \underline{\boldsymbol{G}}_J$
- Useful wrt. derived quantities like strain etc.

Characteristics of the motion

- $\underline{\boldsymbol{x}} = \underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, \tau)$ where $\underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, \tau)$ is called the motion of $\mathcal{B}$

- we demand that the motion has to be continuous (smooth)

$$\underline{\boldsymbol{x}} = \underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, \tau) \qquad \Longleftrightarrow \qquad \underline{\boldsymbol{X}} = \underline{\boldsymbol{\chi}}^{-1}(\underline{\boldsymbol{x}}, \tau)$$

- velocity of a material point with label $\underline{\boldsymbol{X}}$ at $\tau = t$

$$\underline{\boldsymbol{v}}(\underline{\boldsymbol{X}}, \tau = t) = \lim_{\Delta t \to 0} \frac{\underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, t + \Delta t) - \underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, t)}{\Delta t} =: \frac{\mathrm{d}}{\mathrm{d}\tau} \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, \tau)\Big|_{\tau = t} = \dot{\underline{\boldsymbol{x}}}(t)$$

also called the material time derivative[2]

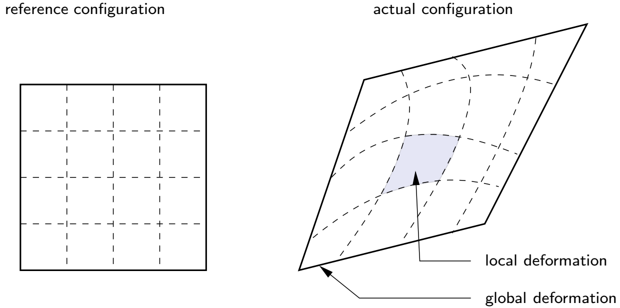
[2] Revise your notes wrt. material / spatial time derivative!

Kinematics

How to quantify the deformation of a body?

Definition (Deformation)

Deformation means change in volume and shape.



Gradient of the motion

Let's focus on the local deformation!

- How to derive deformation measures from the motion of $\mathcal{B}$?
- Motion of one material point doesn't say anything about deformation.
- Difference between the motion of two neighbouring material points $\Rightarrow$ gradient of the motion $\boldsymbol{F}$.
- $\boldsymbol{F}$ usually called "deformation gradient" for historical reasons.

Remember that $\underline{\boldsymbol{x}} = \underline{\boldsymbol{\chi}}(\underline{\boldsymbol{X}}, \tau)$ or in lazy notation $\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}}(\underline{\boldsymbol{X}}, \tau)$

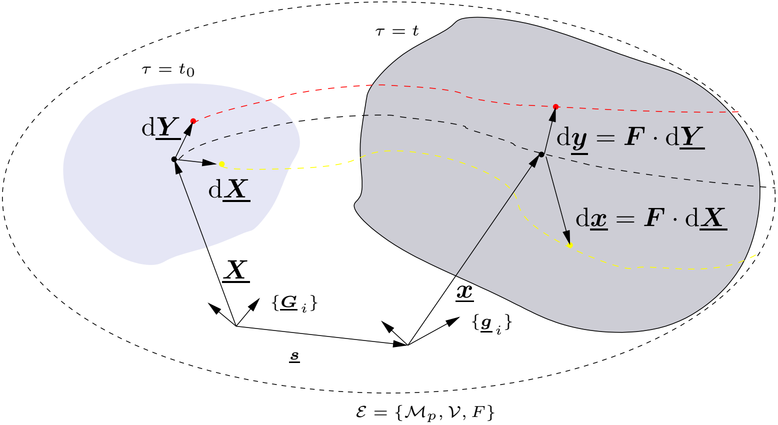
- How changes $\underline{\boldsymbol{x}}$ due to a change of $\underline{\boldsymbol{X}}$ at a fixed time $\tau = t$?
- $\underline{\boldsymbol{x}}$ is a vector field which depends on the position vector $\underline{\boldsymbol{X}}$
- Gradient[3] of $\underline{\boldsymbol{x}}$ wrt. $\underline{\boldsymbol{X}}$ $\Rightarrow$ 2^{nd} order tensor called $\boldsymbol{F}$

$$\boldsymbol{F} := \mathrm{Grad}\,\underline{\boldsymbol{x}} = \frac{\partial x^i}{\partial X^K}\,\underline{\boldsymbol{g}}_i \otimes \underset{\sim}{\boldsymbol{G}}^K$$

- 2^{nd} order tensors are linear mappings, hence[4]

$$\mathrm{d}\underline{\boldsymbol{x}} = \boldsymbol{F} \cdot \mathrm{d}\underline{\boldsymbol{X}}$$

[3] Revise lecture notes wrt. Fréchet differentiability!
[4] Revise lecture notes wrt. Gateaux differential!



alternative notations:

$$\boldsymbol{F} = \mathrm{Grad}\,\underline{\boldsymbol{x}} = \mathrm{grad}_{\underline{\boldsymbol{X}}}\,\underline{\boldsymbol{x}} = \underline{\boldsymbol{x}} \otimes \underline{\boldsymbol{\nabla}}_{\underline{\boldsymbol{X}}} \qquad \left(\underline{\boldsymbol{\nabla}}_{\underline{\boldsymbol{X}}} = \frac{\partial}{\partial X^k}\,\underset{\sim}{\boldsymbol{G}}^k\right)$$

Definition (Push forward / Pull back of vectors)

push forward	pull back
$\mathrm{d}\underline{\boldsymbol{x}} = \boldsymbol{F} \cdot \mathrm{d}\underline{\boldsymbol{X}}$	$\mathrm{d}\underline{\boldsymbol{X}} = \boldsymbol{F}^{-1} \cdot \mathrm{d}\underline{\boldsymbol{x}}$

Theorem (Polar decomposition theorem)

$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{R}$
with

- $\boldsymbol{R}, \boldsymbol{U}, \boldsymbol{V}$ *unique*
- $\boldsymbol{U}, \boldsymbol{V}$ *right / left Cauchy-Green tensor (pure stretch)*
- $\boldsymbol{R}$ *pure rotation* $\boldsymbol{R} \cdot \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{I}$

Strain measures

The aim is to quantify deformation under the conditions that

- to quantify means to compute numbers,
- "0.0" should mean "no deformation",
- any measure should converge to the same limit in the case of small deformation,
- measures should be objective.

Facts:

- $\boldsymbol{F}$ is not a suitable measure, just because it contains not only stretch but also pure rotation.
- $\boldsymbol{F}$ can be used to define proper strain measures.

Commonly used strain measures:

name	symbolic formula	index notation
Green-Lagrange[5]	$\boldsymbol{E} = \frac{1}{2}[\boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} - \boldsymbol{I}]$	$\boldsymbol{E} = E_{IJ} \underset{\sim}{\boldsymbol{G}}^{I} \otimes \underset{\sim}{\boldsymbol{G}}^{J}$
Euler-Almansi[6]	$\boldsymbol{A} = \frac{1}{2}[\boldsymbol{I} - \boldsymbol{F}^{-\mathrm{T}} \cdot \boldsymbol{F}^{-1}]$	$\boldsymbol{A} = A_{ij} \underset{\sim}{\boldsymbol{g}}^{i} \otimes \underset{\sim}{\boldsymbol{g}}^{j}$

There are more.

[5] for a detailed interpretation of the coordinates of $\boldsymbol{E}$ see homework 8
[6] for a detailed interpretation of the coordinates of $\boldsymbol{A}$ see exercise 5

Deformation velocity

Spatial velocity gradient $\boldsymbol{L}$ $\qquad (\mathrm{d}\underline{\boldsymbol{x}})^\bullet = \boldsymbol{L} \cdot \mathrm{d}\underline{\boldsymbol{x}}$

$$\boldsymbol{L} = \mathrm{grad}\, \underline{\boldsymbol{v}} = \underline{\boldsymbol{v}} \otimes \underline{\boldsymbol{\nabla}}_{\underline{\boldsymbol{x}}} \qquad \text{with} \qquad \underline{\boldsymbol{\nabla}}_{\underline{\boldsymbol{x}}} = \frac{\partial}{\partial x^k} \underset{\sim}{\underline{\boldsymbol{g}}}^k$$

Decomposition

$$\boldsymbol{L} = \underbrace{\frac{1}{2}[\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}]}_{\boldsymbol{D}} + \underbrace{\frac{1}{2}[\boldsymbol{L} - \boldsymbol{L}^{\mathrm{T}}]}_{\boldsymbol{W}}$$

$\boldsymbol{D}$ stretch velocity tensor, $\boldsymbol{W}$ spin tensor

Balance equations

Why balance equations?

- So far,
  - we can observe the motion and measure the local deformation, respectively its velocity etc., **but**
  - we still don't have any concept to express a cause.
- Obviously, what happens with $\mathcal{B}$ must have something to do with the interaction of $\mathcal{B}$ with the world.
- A body which doesn't interact maintains its ...[7]
- Interaction is usually formulated via global balance equations.

[7] Take this as a starting point and play around with your thoughts.

Global balances

Global balances formulated for the actual configuration

- mass balance

$$\dot{\mathfrak{M}} = \frac{\mathrm{d}}{\mathrm{d}\tau} \int\limits_{\mathcal{B}} \rho \, \mathrm{d}V = 0$$

- energy balance

$$\dot{U} + \dot{K} = \dot{W} + \dot{Q}$$

with $\dot{E} = \dot{U} + \dot{K}$ rate of energy change of the body
$\dot{Q}$ heat extraction / supply rate
$\dot{W}$ extraction / supply of mechanical power

Global energy balance in detail

Energy of the body

$$E = \int_{\mathcal{B}} \rho\, e \,\mathrm{d}V + \int_{\mathcal{B}} \frac{1}{2} \left\langle \boldsymbol{M} \cdot \underline{\boldsymbol{v}}, \underline{\boldsymbol{v}} \right\rangle \mathrm{d}V$$

Exchange rates

$$\dot{W} = \int_{\mathcal{B}} \left\langle \underset{\sim}{\boldsymbol{f}}, \underline{\boldsymbol{v}} \right\rangle \mathrm{d}V \qquad + \qquad \int_{\partial\mathcal{B}} \left\langle \underset{\sim}{\boldsymbol{t}}, \underline{\boldsymbol{v}} \right\rangle \mathrm{d}A$$

$$\dot{Q} = \int_{\mathcal{B}} r \,\mathrm{d}V \qquad + \qquad \int_{\partial\mathcal{B}} h \,\mathrm{d}A$$

with $\boldsymbol{M} = \rho\, \delta_{ij}\, \underset{\sim}{\boldsymbol{g}}^{i} \otimes \underset{\sim}{\boldsymbol{g}}^{j}$

Stress concept

We consider $\int\limits_{\partial\mathcal{B}} \left\langle \underset{\sim}{\boldsymbol{t}}, \underline{\boldsymbol{v}} \right\rangle \, \mathrm{d}A = \int\limits_{\partial\mathcal{B}} \underset{\sim}{\boldsymbol{t}} \cdot \underline{\boldsymbol{v}} \, \mathrm{d}A$

Reasoning which leads to this term:

- integral must be power (work per time)
- power is an energetic quantity
  - we see force quantities as covectors ($\in \mathcal{V}^*$)
  - pairing $<$ covector, vector $>$ gives an energetic quantitiy
- we start with $\underline{\boldsymbol{v}} \in \mathcal{V}$ because we need power $\Rightarrow \underset{\sim}{\boldsymbol{t}} \in \mathcal{V}^*$
- $\underset{\sim}{\boldsymbol{t}}$ must be a force per unit area $\Rightarrow$ "surface traction"

Stress concept:

- $\underset{\sim}{\boldsymbol{t}}$ means a force flux via the outer surface of $\mathcal{B}$
- fluxes depend in general on the properties of the surface, i.e.

$$\underset{\sim}{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{t}}(\underset{\sim}{\boldsymbol{n}}, \underset{\sim}{\boldsymbol{n}} \otimes \underset{\sim}{\boldsymbol{n}}, ...)$$

- **assumption:** $\underset{\sim}{\boldsymbol{t}}$ depends only linearly on $\underset{\sim}{\boldsymbol{n}}$

$$\rightsquigarrow \quad \underset{\sim}{\boldsymbol{t}} = \underset{\sim}{\boldsymbol{n}} \cdot \boldsymbol{\sigma}$$

- $\boldsymbol{\sigma} = \sigma^{k}_{\ p}\, \underline{\boldsymbol{g}}_{\,k} \otimes \underset{\sim}{\boldsymbol{g}}^{\,p}$ is called the Cauchy stress tensor
- other stress measures[8] can be derived from $\boldsymbol{\sigma}$

[8]see handout “global balances” and exercise 7

Local balance equations

Local balances from global balances in general

- pull back the integrals to which a material time derivative is applied to the reference configuration
- perform the time derivative by changing the order of integration and differentiation
- push forward to the actual configuration
- apply divergence theorem to surface integrals
- stipulate that the balance has to be fulfilled for every arbitrary part of the body $\Rightarrow$ local balance

Strategy used to derive local balances from global balances:

Global Balances	leads to local balance of
1. mass	$\Rightarrow$ mass
2. energy	$\Rightarrow$ energy
3. Invariance of the energy balance under superimposed rigid motion	
• rigid translation $\rightarrow$ reduced energy balance	$\Rightarrow$ momentum
• rigid rotation applied to the reduced energy balance	$\Rightarrow$ "angular momentum" (symmetry Cauchy-stress)

Local balances

actual configuration		reference configuration
$\dot{\rho} + \rho \mathsf{div}(\underline{\boldsymbol{v}}) = 0$	**mass**	$\dot{\rho}_0 = 0$
$\boldsymbol{M} \cdot \underline{\boldsymbol{v}} = \mathsf{div}(\boldsymbol{\sigma}^{\mathrm{T}}) + \underset{\sim}{\boldsymbol{f}}$	**momentum**	$\boldsymbol{M}_0 \cdot \underline{\boldsymbol{v}} = \mathsf{Div}(\boldsymbol{F} \cdot\ ^{II}\boldsymbol{T}) + \underset{\sim}{\boldsymbol{f}}_0$
$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$	**angular momentum**	$^{II}\boldsymbol{T}^{\mathrm{T}} =\ ^{II}\boldsymbol{T}$
$\rho \dot{e} = \boldsymbol{\sigma} \cdot \cdot \boldsymbol{D} + r - \mathsf{div}(\underline{\boldsymbol{q}})$	**energy**	$\rho_0 \dot{e} =\ ^{II}\boldsymbol{T} \cdot \cdot \dot{\boldsymbol{E}} + r_0 - \mathsf{Div}(\underline{\boldsymbol{q}}_0)$

Alternative approach

Local balances of momentum and angular momentum usually from

- global balance of momentum

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{B}} \boldsymbol{M} \cdot \underline{\boldsymbol{v}} \, \mathrm{d}V = \int_{\mathcal{B}} \underset{\sim}{\boldsymbol{f}} \, \mathrm{d}V + \int_{\partial\mathcal{B}} \underset{\sim}{\boldsymbol{t}} \, \mathrm{d}A$$

- global balance of angular momentum

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_{\mathcal{B}} \underline{\boldsymbol{x}} \times (\boldsymbol{M} \cdot \underline{\boldsymbol{v}}) \, \mathrm{d}V = \int_{\mathcal{B}} \underline{\boldsymbol{x}} \times \underset{\sim}{\boldsymbol{f}} \, \mathrm{d}V + \int_{\partial\mathcal{B}} \underline{\boldsymbol{x}} \times \underset{\sim}{\boldsymbol{t}} \, \mathrm{d}A$$

Problem: severe conceptual difficulties in the course of the derivation
"Remedy": abandon vector / covector setting and decide which kind of vectors should be used to form a "scalar product"
$\Rightarrow$ suspicious!

Constitutive equations

Number of unknowns versus number of equations so far

Consider the isothermal case in terms of the reference configuration

$$M_0 \cdot \underbrace{\dot{\underline{\boldsymbol{v}}}}_{\ddot{\underline{\boldsymbol{U}}}} = \mathrm{Div}\big({}^{II}\boldsymbol{T} \cdot \underbrace{\boldsymbol{F}}_{\underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_{\underline{\boldsymbol{X}}} + \boldsymbol{I}} \big) - \underset{\sim}{\boldsymbol{f}} \qquad \mathbf{3}$$

$$\boldsymbol{E} = \frac{1}{2}\Big[\underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_{\underline{\boldsymbol{X}}} + \big[\underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_{\underline{\boldsymbol{X}}}\big]^{\mathrm{T}} + \big[\underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_{\underline{\boldsymbol{X}}}\big]^{\mathrm{T}} \cdot \big[\underline{\boldsymbol{U}} \otimes \boldsymbol{\nabla}_{\underline{\boldsymbol{X}}}\big]\Big] \qquad \mathbf{6}$$

boundary conditions

$$\underline{\boldsymbol{U}}(\underline{\boldsymbol{X}}, \tau) = \bar{\underline{\boldsymbol{U}}}(\underline{\boldsymbol{X}}, \tau) \quad \underline{\boldsymbol{X}} \in \partial \mathcal{B}_{0_v}$$
$$\underset{\sim}{\boldsymbol{T}}(\underline{\boldsymbol{X}}, \tau) = \bar{\underset{\sim}{\boldsymbol{T}}}(\underline{\boldsymbol{X}}, \tau) \quad \underline{\boldsymbol{X}} \in \partial \mathcal{B}_{0_T}$$

initial conditions, e.g

$$\bar{\underline{\boldsymbol{U}}}(\underline{\boldsymbol{X}}, \tau = t_0) = \bar{\underline{\boldsymbol{U}}}^{t_0}(\underline{\boldsymbol{X}})$$
$$\dot{\underline{\boldsymbol{U}}}(\underline{\boldsymbol{X}}, \tau = t_0) = \dot{\underline{\boldsymbol{U}}}^{t_0}(\underline{\boldsymbol{X}})$$

$$\left.\begin{array}{lll} \underline{\boldsymbol{U}}(\underline{\boldsymbol{X}}, \tau) & 3 & \\ {}^{II}\boldsymbol{T}(\underline{\boldsymbol{X}}, \tau) & 6 & ({}^{II}\boldsymbol{T} = {}^{II}\boldsymbol{T}^{\mathrm{T}}) \\ \boldsymbol{E}(\underline{\boldsymbol{X}}, \tau) & 6 & (\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}}) \end{array}\right\}$$

15 unknowns
only 9 eqns.

$\Rightarrow$ ${}^{II}\boldsymbol{T} \Leftrightarrow \boldsymbol{E}$ missing!

Clausius-Duhem inequality

Globally:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\mathcal{B}} \rho\, s\, \mathrm{d}V \geq \int\limits_{\mathcal{B}} \frac{r}{\Theta}\, \mathrm{d}V - \int\limits_{\partial \mathrm{B}} \frac{1}{\Theta} \left\langle \underset{\sim}{\boldsymbol{n}}, \underline{\boldsymbol{q}} \right\rangle \mathrm{d}A$$

locally in terms of the reference configuration[9]:

$$\rho_0 \dot{s} \geq \frac{1}{\Theta} r_0 - \mathsf{Div}\left(\frac{1}{\Theta}\, \underline{\boldsymbol{q}}_0\right)$$

entropy s and absolute temperature Θ as primitive concepts

Postulate: constitutive equations must obey Cl.-Duhem ineq.

[9] $r_0 = J\, r$, $\underline{\boldsymbol{q}}_0 = J \underline{\boldsymbol{q}} \cdot \boldsymbol{F}^{-\mathrm{T}}$

Hyperelasticity

analogy to classical P-V-Θ-thermodynamics

$$e = e(s, \boldsymbol{E})$$

entropy s cannot be measured directly but temperature can be

$$\Psi := e - \Theta s \qquad \Rightarrow \qquad \Psi = \Psi(\Theta, \boldsymbol{E})$$

One can show[10] that

$$^{II}\boldsymbol{T} = \rho_0 \frac{\partial \Psi}{\partial \boldsymbol{E}} \quad , \quad s = -\frac{\partial \Psi}{\partial \Theta} \quad \text{and} \quad -\frac{1}{\Theta^2} \mathsf{Grad}\Theta \cdot \underline{\boldsymbol{q}}_0 \geq 0$$

must hold in order to fulfil CDI.

[10] see your notes or standard text books

Advantage: everything comes down to Ψ

Most general form for an isotropic, hyperelastic material

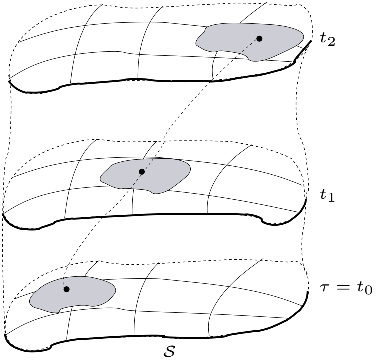
$$\Psi = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{ijk} \left[I_1(\boldsymbol{C}) - 3\right]^i \, \left[I_2(\boldsymbol{C}) - 3\right]^j \, \left[I_3(\boldsymbol{C}) - 3\right]^k$$

with $\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F}$ and its invariants I_1, I_2, I_3

remember: $\boldsymbol{E} = \frac{1}{2}[\boldsymbol{C} - \boldsymbol{I}]$

Keep in mind that the α_{ijk} are material parameters which have to be determined experimentally!

A last slide - imagine space-time like this



- space no longer euclidean $\Rightarrow$ no global vector balances anymore
- space concept - differentiable manifold
- body differentiable manifold too
- vector calculus now only locally using so-called tangent space
- new tools appear almost naturally

Problem as such appeared in Theoretical Physics **but** the tools developed to deal with it are most fruitful even for standard continuum mechanics.